

# HOCHSCHILD COHOMOLOGY OF DEFORMATION QUANTIZATIONS OVER $\mathbb{Z}/p^n\mathbb{Z}$

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**ABSTRACT.** Let  $A_1$  be an Azumaya algebra over a smooth affine symplectic variety  $X$  over  $\text{Spec } F_p$ , where  $p$  is an odd prime. Let  $A$  be a deformation quantization of  $A_1$  over the  $p$ -adic integers. In this note we show that for all  $n \geq 1$ , the Hochschild cohomology of  $A/p^n A$  is isomorphic to the de Rham-Witt complex  $W_n\Omega_X^*$  of  $X$  over  $\mathbb{Z}/p^n\mathbb{Z}$ . We also compute the center of deformations of certain affine Poisson varieties over  $F_p$ .

Let  $\mathbf{k}$  be a perfect field of characteristic  $p > 2$ . For  $n \geq 1$ , let  $W_n(\mathbf{k})$  denote the ring of length  $n$  Witt vectors over  $\mathbf{k}$ . Also,  $W(\mathbf{k})$  will denote the ring of Witt vectors over  $\mathbf{k}$ . Let  $X$  be an affine smooth symplectic variety over  $\mathbf{k}$ . Let  $\{, \}$  denote the corresponding Poisson bracket on  $\mathcal{O}_X$ , the structure ring of  $X$ . Let  $A_1$  be an Azumaya algebra over  $X$  (equivalently over  $\mathcal{O}_X$ .) Thus, we may (and will) identify the center of  $A_1$  with  $\mathcal{O}_X : Z(A_1) = \mathcal{O}_X$ . A deformation quantization of  $A_1$  over  $W(\mathbf{k})$  is, by definition, a flat associative  $W(\mathbf{k})$ -algebra  $A$  equipped with an isomorphism  $A/pA \simeq A_1$  such that for any  $a, b \in A$  such that  $a \bmod p \in \mathcal{O}_X, b \bmod p \in \mathcal{O}_X$ , one has

$$\{a \bmod p, b \bmod p\} = \left(\frac{1}{p}[a, b]\right) \bmod p.$$

As usual, for an associative algebra  $B$  its Hochschild cohomology will be denoted by  $HH^*(B)$ . Also, for a commutative ring  $S$  over  $\mathbf{k}$ ,  $W_n\Omega_S^*$  will denote the de Rham-Witt complex of  $S$  over  $W_n(\mathbf{k}), n \geq 1$ . In the above notation, our main result is the following

**Theorem 1.** *Let  $A_1$  be an Azumaya algebra over an affine symplectic variety  $X$  over  $\mathbf{k}$ . Let  $A$  be a deformation quantization of  $A_1$  over  $W(\mathbf{k})$ . Then for all  $n \geq 1$  we have a canonical isomorphism of graded algebras  $HH^*(A/p^n A) \simeq W_n\Omega_{\mathcal{O}_X}^*$ .*

This isomorphism on the level of centers was obtained (in more general form) by Stewart and Vologodsky [SV].

Before proving the result, we will recall Bockstein operations on the Hochschild cohomology of algebras over  $W_m(\mathbf{k})$ .

Let  $(S, \delta)$  be a flat differential graded algebra over  $W(\mathbf{k})$ . We set  $S_n = S/p^n S, n \geq 1$ . Then we have a map  $v^l : S_n \rightarrow S_{n+l}$  defined as follows:

$v^l(z) = p^l \tilde{z} \bmod p^{n+l}S$ , where  $\tilde{z}$  is a lift of  $z$  in  $S$ . Clearly  $v^l$  is well-defined. Also we will denote the quotient map  $S_n \rightarrow S_{n-l}$  by  $r^l$ . Clearly  $v^l, l \geq 1$  are maps of complexes, while  $r^l$  is a homomorphism of differential graded algebras. We have the following short exact sequence of complexes

$$0 \rightarrow S_n \xrightarrow{v^n} S_{2n} \xrightarrow{r^n} S_n \rightarrow 0.$$

Denote by  $\bar{v}^l : H^*(S_n) \rightarrow H^*(S_{n+l}), \bar{r}^l : H^*(S_n) \rightarrow H^*(S_{n-l})$  the maps induced on cohomologies by  $v^l : S_n \rightarrow S_{n+l}$  and  $r^l : S_n \rightarrow S_{n-l}$  respectively.

Thus we have the following long exact sequence

$$\cdots \rightarrow H(S_n) \xrightarrow{\bar{v}^n} H(S_{2n}) \xrightarrow{\bar{r}^n} H(S_n) \xrightarrow{d_n} H(S_n) \rightarrow \cdots,$$

here  $d_n : H(S_n) \rightarrow H(S_n)$  denotes the connecting homomorphism. The following lemma is well-known and straightforward

**Lemma 2.** *The algebra  $(H^*(S_n), d_n)$  is a differential graded algebra in which the following identities hold*

$$\begin{aligned} \bar{r}\bar{v} &= \bar{v}\bar{r} = p, & d_n\bar{r} &= p\bar{r}d_n, & \bar{r}d_n\bar{v} &= d_n, & \bar{v}d_n &= pd_n\bar{v}, \\ x\bar{v}(y) &= \bar{v}(\bar{r}(x)y), & \bar{v}(xd_ny) &= \bar{v}(x)d_n(\bar{v}y). \end{aligned}$$

Let  $A$  be an associative flat  $W(\mathbf{k})$ -algebra, not necessarily satisfying assumptions of Theorem 1. For  $n \geq 1$ , set  $A_n = A/p^n A$ . Recall that the standard Hochschild complex  $S = (\bigoplus_n C^*(A_n, A_n), \delta) = (\text{Hom}_{W_n(\mathbf{k})}(A_n^{\otimes*}, A_n), \delta)$  is a differential graded algebra under the cup product. Thus we may apply the above constructions to  $S$ . Hence we have maps

$$\begin{aligned} \bar{r} : HH^*(A_n) &\rightarrow HH^*(A_{n-1}), & \bar{v} : HH^*(A_n) &\rightarrow HH^*(A_{n+1}) \\ d : HH^*(A_n) &\rightarrow HH^{*+1}(A_n) \end{aligned}$$

satisfying the identities from Lemma 2.

The center of the algebra  $A_n$  will be denoted by  $Z_n$ . Stewart and Volodsky [[SV], formula (1.3)] constructed a  $W_n(\mathbf{k})$ -algebra homomorphism  $\phi_n$  from the ring of length  $n$  Witt vectors over  $Z_1$  to  $Z_n$ , defined as follows: given  $(z_1, \dots, z_n) \in W_n(Z_1), z_i \in Z_1, 1 \leq i \leq n$ , define

$$\phi_n(z_1, \dots, z_n) = \sum_{i=1}^n p^{i-1} \tilde{z}_i p^{n-i}$$

where  $\tilde{z}_i$  is a lift of  $z_i$  in  $A$ . They checked that  $\phi_n$  is well-defined and

$$V\phi_n = \phi_n\bar{v}, \quad F\phi_n = \phi_n\bar{r},$$

where

$$V : W_n(Z_1) \rightarrow W_{n+1}(Z_1), \quad F : W_n(Z_1) \rightarrow W_{n-1}(Z_1)$$

are Verschiebung, respectively Frobenius maps on the ring of Witt vectors of  $Z_1$ .

For  $z \in Z_n$ , denote by  $\underline{z} = \tilde{z}^p \bmod p^{n+1}A, \underline{z} \in Z_{n+1}$ , where  $\tilde{z}$  is a lift of  $z$  in  $A$ . It was checked in [SV] that  $\underline{z}$  is independent of the choice of  $\tilde{z}$ . Given

an element  $a \in Z_1$ , we will denote by  $\underline{a}$  the Teichmuller lift of  $a$  in  $W_n(Z_1)$ . Thus  $\phi_n(\underline{a}) = \tilde{a}^{p^n} \in Z_n$ , where  $\tilde{a}$  is a lift of  $a$  in  $A_n$ .

It follows from [SV, Lemma 2.5] that

$$\bar{r}(d(\underline{z})) = z^{p-1}d(z), z \in Z_n.$$

The latter equality, combined with Lemma 2 and the construction of the De Rham-Witt complex  $W_n\Omega^*$  [LZ] implies that for all  $n \geq 1$ , there is the unique homomorphism of differential graded algebras

$$\phi_n^* : W_n\Omega_{Z_1}^* \rightarrow HH^*(A_n)$$

such that

$$\bar{v}\phi_n^* = \phi_{n+1}^*V, \quad \bar{r}\phi_{n+1}^* = \phi_n^*F,$$

where

$$F : W_n\Omega_{Z_1}^* \rightarrow W_{n-1}\Omega_{Z_1}^*, \quad V : W_n\Omega_{Z_1}^* \rightarrow W_{n+1}\Omega_{Z_1}^*$$

denote the Frobenius, respectively Verschiebung maps in the de Rham-Witt complex of  $Z_1$ . We can now prove the main result in this note.

*Proof of Theorem 1.* Let  $A$  be a deformation quantization of an Azumaya algebra  $\tilde{A}$  over  $W(\mathbf{k})$ . Thus we may (and will) identify  $\tilde{A}$  with  $A_1 = A/pA$  and  $\mathcal{O}_X$  with  $Z_1$ . We will prove by induction on  $n$  that the map  $\phi_n$  constructed above is an isomorphism. Let  $n = 1$ . Since by the assumption  $A_1$  is an Azumaya algebra over  $Z_1 = Z(A_1)$ , it follows that  $HH^*(A_1) \simeq HH^*(Z_1)$  ([Sc]). On the other hand since  $\text{Spec } Z_1$  is a symplectic variety over  $\mathbf{k}$ , the Hochschild cohomology  $HH^*(Z_1)$  is isomorphic to  $\Omega_{Z_1}^*$  by the Hochschild-Kostant-Rosenberg theorem. Moreover, it is easy to check that given  $fdg \in \Omega_{Z_1}^1$ , we have

$$\phi_1(fdg) = f\{g, -\} \in \text{Der}(A_1),$$

which agrees with the Hochschild-Kostant-Rosenberg isomorphism. Thus,  $\phi_1$  is an isomorphism.

Let  $n \geq 1$ . Assume that  $\phi_i$  is an isomorphism for all  $i \leq n$ . We claim that the connecting homomorphism  $\delta_n : HH^*(A_n) \rightarrow HH^{*+1}(A_1)$  between Hochschild cohomologies arising from the exact sequence of  $A_{n+1}$ -bimodules

$$0 \rightarrow A_1 \xrightarrow{v^n} A_{n+1} \xrightarrow{r} A_n \rightarrow 0$$

equals  $\bar{r}^n d$ . Indeed, it follows directly from commutativity of the following diagram

$$\begin{array}{ccccccc} HH^*(A_n) & \xrightarrow{\bar{v}^n} & HH^*(A_{2n}) & \xrightarrow{\bar{r}^n} & HH^*(A_n) & \xrightarrow{d} & HH^{*+1}(A_n) \\ \downarrow \bar{r}^{n-1} & & \downarrow \bar{r}^{n-1} & & \downarrow id & & \downarrow \bar{r}^{n-1} \\ HH^*(A_1) & \xrightarrow{\bar{v}^n} & HH^*(A_{n+1}) & \xrightarrow{\bar{r}} & HH^*(A_n) & \xrightarrow{\delta_n} & HH^{*+1}(A_1). \end{array}$$

We have the following commutative diagram of long exact sequences

$$\begin{array}{ccccccc}
W_1\Omega_{Z_1}^* & \xrightarrow{V^n} & W_{n+1}\Omega_{Z_1}^* & \xrightarrow{F} & W_n\Omega_{Z_1}^* & \xrightarrow{F^{n-1}d} & W_1\Omega_{Z_1}^{*+1} \\
\downarrow \phi_1 & & \downarrow \phi_{n+1} & & \downarrow \phi_n & & \downarrow \phi_1 \\
HH^*(A_1) & \xrightarrow{\bar{v}^n} & HH^*(A_{n+1}) & \xrightarrow{\bar{r}} & HH^*(A_n) & \xrightarrow{\bar{r}^{n-1}d} & HH^{*+1}(A_1)
\end{array}$$

Here exactness of the top sequence is a well-known property of the de Rham-Witt complex [Il, Proposition 3.11].

Thus, it follows from the inductive assumption that  $\phi_{n+1}$  is an isomorphism. This concludes the proof of Theorem 1.  $\square$

Given a smooth affine variety  $Y$  over a ring  $S$ , we will denote the ring of crystalline (or PD) differential operators on  $Y$  by  $D_Y = D_{Y/S}$ .

**Corollary 3.** *Let  $Y$  be a smooth affine variety over  $W(\mathbf{k})$ . For  $n \geq 1$ , let  $Y_n$  denote the mod  $p^n$  reduction of  $Y$ . Then  $HH^*(D_{Y_n})$  is isomorphic to  $W_n\Omega_{T_{Y_1}^*}^*$ , where  $T_{Y_1}^*$  is the cotangent bundle of  $Y_1$ .*

*Proof.* Put  $A = D_{\tilde{Y}}$ . Then  $A_n = A/p^n A = D_{Y_n}$ ,  $n \geq 1$ . It is well-known that  $\tilde{A} = D_{Y_1}$  is an Azumaya algebra over the Frobenius twist of  $T_{Y_1}^* = X$ -the cotangent bundle of  $Y_1$  [BMR]. Also, the corresponding Poisson bracket on  $X$  is symplectic by [BK]. Thus, Theorem 1 applies directly.  $\square$

Next we will establish a generalization of a result by Stewart and Vologodsky [[SV], Theorem 1]. We will follow their proof very closely.

**Theorem 4.** *Let  $(X, \{\cdot, \cdot\})$  be an affine Poisson normal variety over  $\mathbf{k}$  such that it has an open symplectic leaf. Let  $A$  be a flat associative  $W_n(\mathbf{k})$ -algebra, such that  $Z(A/pA) = \mathcal{O}_X$  and the deformation Poisson bracket of  $Z(A/pA)$  coincides with the bracket of Poisson variety  $X$ . Then the map  $\phi_m : W_m(\mathcal{O}_X) \rightarrow Z(A/p^m A)$  defined above is an isomorphism for all  $1 \leq m \leq n$ .*

*Proof.* We will proceed by induction on  $m$ . For  $m = 1$  there is nothing to prove. As before, we will put  $A_m = A/p^m A$ ,  $Z_m = Z(A_m)$ . Recall also the isomorphisms  $v^i : A_m \rightarrow p^i A_{m+i}$  defined by multiplying a lift in  $A_{m+i}$  by  $p^i$ ,  $i \leq n - m$ . The following is the key step (as in [SV], Lemma 2.4).

**Lemma 5.** *Let  $\tau : W_m(Z_1) \rightarrow \text{Der}(Z_1, Z_1)$  be defined as follows*

$$\tau(z_1, \dots, z_m) = \sum_{i=1}^m z_i^{p^{m-i}-1} \{z_i, -\}.$$

*Then  $\tau(z_1, \dots, z_m) = 0$  if and only if  $z_i = a_i^p$  for some  $a_i \in Z_1$ .*

*Proof.* Let  $f \in Z_1$  be such that  $U = \text{Spec}(Z_1)_f$  is an open symplectic leaf of  $X$ . Put  $S = (Z_1)_f$ , let  $\omega$  be the symplectic form of  $U$ . Transferring the equality  $\tau(z_1, \dots, z_n) = 0$  via the isomorphism  $\omega : \text{Der}_{\mathbf{k}}(S, S) \simeq \Omega_S^1$  to  $\Omega_S^1$ , we obtain

$$\sum_{i=1}^m z_i^{p^{m-i}-1} dz_i = 0.$$

Recall that the inverse Cartier map  $C^{-1} : \Omega_S^1 \rightarrow \Omega_S^1/dS$ , which is defined as follows  $C^{-1}(fdg) = f^p g^{p-1} dg$ , is an injective homomorphism onto  $H^1(\Omega_S^*) \subset \Omega_S^1/dS$ . Thus, we obtain

$$C^{-1}\left(\sum_{i < m} z_i^{p^{m-i}-1} dz_i\right) = 0 \in \Omega^1/dS$$

Hence by injectivity of  $C^{-1}$  we conclude that

$$\sum_{i=1}^{m-1} z_i^{p^{m-1-i}-1} dz_i = 0 \in \Omega_S^1.$$

Using induction assumption, we conclude that  $z_i \in S^p, i > 1$ . Thus  $dz_1 = 0$ , so  $z_1 \in S^p$ . So, for each  $i, z_i \in Z_1 \cap S^p$ . Since by the assumption  $Z_1$  is a normal domain, we have that  $Z_1 \cap S^p = (Z_1)^p$ . Hence  $z_i \in Z_1^p$  and we are done.  $\square$

Now we can prove the theorem. Let  $x \in Z_m$ . Then by the inductive assumption  $x \bmod p^{m-1} = \phi_{m-1}(z)$  for some  $z \in W_{m-1}(Z_1)$ . It follows from computations in [[SV], proof of Lemma 2.5] that

$$\tau(z) = \left(\frac{1}{p^{m-1}}[\phi_{m-1}(z), -] \bmod p\right)|_{Z_1} = \left(\frac{1}{p^{m-1}}[x, -] \bmod p\right)|_{Z_1} = 0.$$

So by the above lemma,  $z = F(z')$ , for some  $z' \in W_m(Z_1)$ . Hence we conclude that  $\phi_m(z') - x \in p^{m-1}A_m$ . So  $Z_m \subset \phi_m(W_m(Z_1)) + p^{m-1}A_m$ . But

$$Z_m \cap p^{m-1}A_m = v^{m-1}(Z_1) \subset \phi_m(W_m(Z_1)).$$

Therefore  $Z_m = \phi_m(W_m(Z_1))$ .

It remains to check that  $\phi_m$  is injective. Let  $0 \neq x \in W_m(Z_1)$ . Hence we may write  $x = V^i(z)$ , for some  $i$  and  $z \in W_{m-i}(Z_1) \setminus V W_{m-i-1}(Z_1)$ . We have

$$\phi_m(x) = \phi_m(V^i(z)) = v^i(\phi_{m-i}(z)).$$

Clearly  $\phi_i(z) \in A_{m-i} \setminus pA_{m-i}$ . Hence  $\phi_m(V^i(z)) \neq 0$ .  $\square$

The above result can be used to compute centers of large class of algebras over  $W_n(\mathbf{k})$ , including various enveloping algebras of Lie algebras and symplectic reflection algebras.

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